AD 295 505

63-2-3

BSD-TDR-62-375

A THEOREM ON ELECTROMAGNETIC BACKSCATTER

Prepared by

R. J. Wagner and P. J. Lynch

SPACE TECHNOLOGY LABORATORIES, INC.
Redondo Beach, California

6110-7892-RU-000

Contract No. AF 04(694)-1

December 1962

Prepared for

HEADQUARTERS
BALLISTIC SYSTEMS DIVISION
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
Air Force Unit Post Office, Los Angeles 45, California

Prepared: Richard J. Wagner

Richard J. Wagner
Theoretical Physics Department

Prepared: P. James Lynch

P. James Lynch
Theoretical Physics Department

Approved: Saul Altshuler

Saul Altshuler
Manager
Theoretical Physics Department

Approved: R. B. Muchmore

R. B. Muchmore
Director
Physical Research Division

QUANTUM PHYSICS LABORATORY
PHYSICAL RESEARCH DIVISION
SPACE TECHNOLOGY LABORATORIES, INC.
One Space Park
Redondo Beach, California

ABSTRACT

A theorem is proved which gives sufficient conditions under which electromagnetic backscatter from an inhomogeneous object vanishes identically.

TABLE OF CONTENTS

I. INTRODUCTION

The result to be proved may be stated as follows[1]:

If a plane wave is incident along the axis of symmetry of an axially symmetric scatterer, and if the relative permittivity and permeability of the obstacle satisfy the relation $\epsilon(\underset{\sim}{r}) = \mu(\underset{\sim}{r})$, then the radar cross-section is identically zero for all frequencies.

The theorem will first be proved in its general form, then demonstrated for the important special case of an inhomogeneous spherically symmetric scatterer. The analytical methods of the latter derivation will also be used to deduce the angular distribution of low frequency radiation scattered from such a medium. An interesting result at high frequencies will also be pointed out.

II. PROOF OF THE THEOREM

Maxwell's equations, assuming harmonic time dependence, may be written in the form of two stationary wave equations

$$\begin{aligned} \nabla \times \nabla \times \underset{\sim}{E} - k^2 \underset{\sim}{E} + U(\underset{\sim}{r})\underset{\sim}{E} - \frac{\nabla \mu(\underset{\sim}{r})}{\mu(\underset{\sim}{r})} \times \nabla \times \underset{\sim}{E} = 0 \quad , \\ \nabla \times \nabla \times \underset{\sim}{H} - k^2 \underset{\sim}{H} + U(\underset{\sim}{r})\underset{\sim}{H} - \frac{\nabla \epsilon(\underset{\sim}{r})}{\epsilon(\underset{\sim}{r})} \times \nabla \times \underset{\sim}{H} = 0 \quad , \end{aligned} \qquad (1)$$

where the relative permittivity ϵ and relative permeability μ are arbitrary complex functions of $\underset{\sim}{r}$, and $U(\underset{\sim}{r}) \equiv k^2 \left[1 - \mu(\underset{\sim}{r})\, \epsilon(\underset{\sim}{r}) \right]$. The standard boundary conditions for a scattering problem will be assumed: at infinity the total fields are the sum of an incident plane wave and an outgoing spherical wave; the usual continuity conditions at surfaces of discontinuity, if any, of $\epsilon(\underset{\sim}{r})$ and $\mu(\underset{\sim}{r})$ will also be assumed.

It will be convenient to replace the differential equations plus boundary conditions by the two integral equations

$$\underset{\sim}{E}(\underset{\sim}{r}) = \underset{\sim}{E}_o(\underset{\sim}{r}) + \int G(\underset{\sim}{r}, \underset{\sim}{r}') \cdot \left[U(\underset{\sim}{r}')\, \underset{\sim}{E}(\underset{\sim}{r}') - \frac{\nabla' \mu(\underset{\sim}{r}')}{\mu(\underset{\sim}{r}')} \times \nabla' \times \underset{\sim}{E}(\underset{\sim}{r}') \right] d\underset{\sim}{r}'$$

$$\underset{\sim}{H}(\underset{\sim}{r}) = \underset{\sim}{H}_o(\underset{\sim}{r}) + \int G(\underset{\sim}{r}, \underset{\sim}{r}') \cdot \left[U(\underset{\sim}{r}')\, \underset{\sim}{H}(\underset{\sim}{r}') - \frac{\nabla' \epsilon(\underset{\sim}{r}')}{\epsilon(\underset{\sim}{r}')} \times \nabla' \times \underset{\sim}{H}(\underset{\sim}{r}') \right] d\underset{\sim}{r}' \quad , \tag{2}$$

where the tensor Green's function $G(\underset{\sim}{r}, \underset{\sim}{r}')$ is the outgoing solution of

$$\nabla \times \nabla \times G(\underset{\sim}{r}, \underset{\sim}{r}') - k^2\, G(\underset{\sim}{r}, \underset{\sim}{r}') = -\, I \delta\, (\underset{\sim}{r} - \underset{\sim}{r}')$$

with the explicit form

$$G(\underset{\sim}{r}, \underset{\sim}{r}') = (I - \frac{1}{k^2} \nabla \nabla')\, g(\underset{\sim}{r}, \underset{\sim}{r}')$$

$$g(\underset{\sim}{r}, \underset{\sim}{r}') = \frac{e^{ikR}}{-4\pi R} \quad , \qquad \underset{\sim}{R} = \underset{\sim}{r} - \underset{\sim}{r}' \quad .$$

To specialize to the axially symmetric problem, the axis of symmetry will be chosen to be in the direction of propagation of the incident plane wave, i.e., $\underset{\sim}{E}_o \times \underset{\sim}{H}_o^*$ is a vector pointing in the $\hat{k}_o$ direction. The assumption of an outgoing scattered wave implies that, in the backward direction, the phase of $\underset{\sim}{H}$ relative to that of $\underset{\sim}{E}$ has been changed so that $\underset{\sim}{E}_{scatt.} \times \underset{\sim}{H}^*_{scatt.}$ is a vector pointing in the $-\hat{k}_o$ direction. (The change in relative phase of $\underset{\sim}{E}$ and $\underset{\sim}{H}$ is possible since $\underset{\sim}{E}$ and $\underset{\sim}{H}$ are solutions of different equations.)

Suppose now that $\epsilon(\underset{\sim}{r}) \equiv \mu(\underset{\sim}{r})$. The integral equations for $\underset{\sim}{E}$ and $\underset{\sim}{H}$ are then identical, and they may be written as the single integral equation

$$\underset{\sim}{K}(\underset{\sim}{r}) = \underset{\sim}{K}_o(\underset{\sim}{r}) + \int G(\underset{\sim}{r}, \underset{\sim}{r}') \cdot \left[U(\underset{\sim}{r}')\, \underset{\sim}{K}(\underset{\sim}{r}') - \frac{\nabla' \mu}{\mu} \times \nabla' \times \underset{\sim}{K}(\underset{\sim}{r}') \right] d\underset{\sim}{r}' \quad . \tag{3}$$

There are two linearly independent vector solutions to this equation, one corresponding to $\underset{\sim}{K}_o(\underset{\sim}{r})$ polarized in, say, the x-direction and the other corresponding to $\underset{\sim}{K}_o(\underset{\sim}{r})$ polarized in the y-direction (the +z-direction is then the direction of propagation, $\hat{k}_o$, of the incident wave). In the first case the assumed axial symmetry requires the backscattered field to be polarized in the x-direction, while in the second case the backscattered field must be polarized in the y-direction. Furthermore, because of the axial symmetry the phase change of the x-polarized backscattered wave must be exactly equal to the phase change of the y-polarized backscattered wave. Therefore, the relative phase of the two scattered waves is the same as their relative phase in the incident wave. Identifying $\underset{\sim}{E}$ with the solution corresponding to the x-polarized incident wave and $\underset{\sim}{H}$ with the solution corresponding to the y-polarized incident wave, we conclude that $\underset{\sim}{E}_{scatt.} \times \underset{\sim}{H}^*_{scatt.}$ must be a vector pointing in the direction of propagation of the incident wave. But this is consistent with the assumption of outgoing scattered waves only if the backscattered fields are identically zero. There are no explicit restrictions on frequency, and the theorem is therefore valid for all frequencies for which $\epsilon = \mu$.

Note that the strict equality of $\epsilon(\underset{\sim}{r})$ and $\mu(\underset{\sim}{r})$ is actually not necessary for the validity of the theorem. From Eqs. (1), it is clear that the $\underset{\sim}{E}$ and $\underset{\sim}{H}$ equations are identical provided only that $\mu(\underset{\sim}{r}) = b\epsilon(\underset{\sim}{r})$, where b is any constant. (This relation must be satisfied, of course, throughout the whole space containing source and scatterer.) The proof, for $b \neq 1$, proceeds essentially as before, with only a re-definition of the "free-space" wave number required.

III. REMARKS

An interesting consequence of the fact that the $\underset{\sim}{E}$ and $\underset{\sim}{H}$ fields are described by a single vector equation is that there exists an explicit non-differential relation between the $\underset{\sim}{E}$ and $\underset{\sim}{H}$ fields. It may be shown that if $\underset{\sim}{K}(\underset{\sim}{r})$ is a solution to Eq. (3), then $\mathcal{R}\underset{\sim}{K}(\mathcal{R}^{-1}\underset{\sim}{r})$ is also a solution, where $\mathcal{R}$ is the rotation operator (referred to a Cartesian basis)

$$\mathcal{R} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

This may be established by operating on Eq. (3) with $\mathcal{R}$ and replacing the arguments $\underset{\sim}{r}$, $\underset{\sim}{r}'$ of the scalar, vector, and tensor functions by $\mathcal{R}^{-1}\underset{\sim}{r}$, $\mathcal{R}^{-1}\underset{\sim}{r}'$, and using the symmetry relations, $\mu(\mathcal{R}^{-1}\underset{\sim}{r}') = \mu(\underset{\sim}{r}')$, $\mathcal{R}G(\mathcal{R}^{-1}\underset{\sim}{r}, \mathcal{R}^{-1}\underset{\sim}{r}')\mathcal{R}^{-1} = G(\underset{\sim}{r}, \underset{\sim}{r}')$. Since the incident field $\underset{\sim}{K}_0(\underset{\sim}{r})$ is a plane wave propagating in the +z-direction, we may choose $\underset{\sim}{K}_0(\underset{\sim}{r}) = \hat{x}e^{ikz}$. Then $\mathcal{R}\underset{\sim}{K}_0(\mathcal{R}^{-1}\underset{\sim}{r}) = \hat{y}\, e^{ikz}$ is a vector representing an incident plane wave polarized in the +y-direction. Therefore, if $\underset{\sim}{K}(\underset{\sim}{r})$ is one solution of Eq. (3), $\mathcal{R}\underset{\sim}{K}(\mathcal{R}^{-1}\underset{\sim}{r})$ is the second linearly independent solution, and we may identify $\underset{\sim}{E}(\underset{\sim}{r})$ and $\underset{\sim}{H}(\underset{\sim}{r})$ with $\underset{\sim}{K}(\underset{\sim}{r})$ and $\mathcal{R}\underset{\sim}{K}(\mathcal{R}^{-1}\underset{\sim}{r})$, respectively[2].

Incidentally, the relation $\underset{\sim}{H}(\underset{\sim}{r}) = \mathcal{R}\underset{\sim}{E}(\mathcal{R}^{-1}\underset{\sim}{r})$ can now be used to give a very simple proof of the theorem, for on the z-axis $\mathcal{R}\underset{\sim}{E}(\mathcal{R}^{-1}\underset{\sim}{r}) = \mathcal{R}\underset{\sim}{E}(\underset{\sim}{r})$, so that the z-component of $\underset{\sim}{S} = \underset{\sim}{E} \times \underset{\sim}{H}^*$ becomes simply

$$S_z(0,\ 0,\ z) = \left|E_x(0,\ 0,\ z)\right|^2 + \left|E_y(0,\ 0,\ z)\right|^2 \quad .$$

Since $S_z(0, 0, z) \geq 0$, the far-zone scattered field on the symmetry axis must propagate only in the +z-direction, which contradicts the outgoing-wave boundary condition in the backward direction, unless the scattered field is zero.

Since the backscatter cross-section is zero when $\epsilon(\underset{\sim}{r}) = \mu(\underset{\sim}{r})$, it should increase continuously from zero as $\epsilon(\underset{\sim}{r}) - \mu(\underset{\sim}{r})$ is allowed to differ slightly from zero everywhere. This suggests that there may exist an expansion of the fields in terms of a uniformly small quantity, $f\left[\epsilon(\underset{\sim}{r})\right] - f\left[\mu(\underset{\sim}{r})\right]$, which should hold for large, as well as small, values of ϵ. In any case, the fact that the cross-section in the backward direction must vanish when $\epsilon = \mu$, should serve as an additional validity criterion for any approximation method developed to apply when ϵ and μ both differ from unity.

IV. THE SPHERICALLY SYMMETRIC CASE

It would be useful if the angular distribution of the radiation, when $\mu = \epsilon$, could be compared with that when $\epsilon \neq \mu = 1$ in order to determine whether the radiation which is not scattered in the backward direction appears instead at angles close to π, or whether the forward scattering amplitude is enhanced. Such a comparison is not possible for the general case. However, it will now be shown that, for long wavelengths, the angular distribution for a spherically symmetric, but inhomogeneous, scatterer has a particularly simple form when $\epsilon(r) = \mu(r)$, and that the radiation pattern is peaked in the forward direction.

It may be verified by direct substitution into the Maxwell equations that the general solution[3] to the spherically symmetric problem is

$$\epsilon(r)\,\underset{\sim}{E}(\underset{\sim}{r}) = \nabla \times \left[\mu^{\frac{1}{2}}(r)\,\epsilon(r)\,\Psi(r)\,\underset{\sim}{r}\right] + \frac{1}{k}\nabla \times \left\{\nabla \times \left[\epsilon^{\frac{1}{2}}(r)\,\phi(r)\,\underset{\sim}{r}\right]\right\} , \tag{4}$$

$$ik\,\underset{\sim}{B}(\underset{\sim}{r}) = \nabla \times \underset{\sim}{E}(\underset{\sim}{r}) , \tag{5}$$

where Ψ and ϕ satisfy the following equations:

$$\nabla^2 \Psi + \left[k^2 \mu\epsilon - \mu^{\frac{1}{2}} \frac{d^2}{dr^2}(\mu^{-\frac{1}{2}})\right]\Psi = 0 , \tag{6}$$

$$\nabla^2 \phi + \left[k^2 \mu\epsilon - \epsilon^{\frac{1}{2}} \frac{d^2}{dr^2}(\epsilon^{-\frac{1}{2}})\right]\phi = 0 . \tag{7}$$

The boundary conditions on Ψ and ϕ must be such that

$$\underset{\sim}{E}(\underset{\sim}{r}) \xrightarrow[r\to\infty]{} \hat{x}\exp(ikz) + \underset{\sim}{A}(\theta,\phi)\, r^{-1}\exp(ikr) . \tag{8}$$

Here, $\hat{x}$ is the initial polarization, and $\underset{\sim}{A}$ is the vector scattering amplitude. The absolute square of $\underset{\sim}{A}$ is the differential cross section.

The radial equations associated with Eqs. (6) and (7) are

$$\frac{d^2}{dr^2}(rR_\ell) + \left[k^2\mu\epsilon - \mu^{\frac{1}{2}}\frac{d^2}{dr^2}(\mu^{-\frac{1}{2}}) - \frac{\ell(\ell+1)}{r^2}\right] rR_\ell = 0 ,$$

$$\frac{d^2}{dr^2}(rS_\ell) + \left[k^2\mu\epsilon - \epsilon^{\frac{1}{2}}\frac{d^2}{dr^2}(\epsilon^{-\frac{1}{2}}) - \frac{\ell(\ell+1)}{r^2}\right] rS_\ell = 0 ,$$

with boundary conditions

$$rR_\ell\,,\ rS_\ell \xrightarrow[r\to 0]{} 0 ,$$

$$rR_\ell \xrightarrow[r\to\infty]{} \sin(kr - \ell\pi/2 + \delta_\ell) ,$$

$$rS_\ell \xrightarrow[r\to\infty]{} \sin(kr - \ell\pi/2 + \eta_\ell) .$$

The phase shifts, δ_ℓ and η_ℓ, determine the scattering. When $\mu(r) = \epsilon(r)$, the radial equations are identical, and $\delta_\ell = \eta_\ell$.

The scattering amplitude is derived by substituting expansions of the form

$$\sum_{\ell,m} a_{\ell,m} R_\ell(r) Y_\ell^m(\theta, \phi)$$

for ψ and ϕ in Eq. (4). The expansion coefficients can then be evaluated by imposing the asymptotic condition, Eq. (8), provided that the vector plane wave is expressed by its known expansion[4] in spherical harmonics. In general, $\underset{\sim}{A}$ is a complicated function of angles, but because of the equality of the phase shifts δ_ℓ and η_ℓ when $\mu(r) = \epsilon(r)$, considerable simplification of the vector scattering amplitude is possible. It is readily shown that in this case $\underset{\sim}{A}(\theta, \phi)$ reduces to the relatively simple expression

$$\underset{\sim}{A}(\theta, \phi) = (2ik)^{-1} (\cos\phi\, \hat{\theta} - \sin\phi\, \hat{\phi}) \sum_{\ell=1}^{\infty} \frac{(2\ell+1)}{\ell(\ell+1)} (e^{2i\delta_\ell} - 1) \left[(1-\tau) \frac{dP_\ell(\tau)}{d\tau} + \ell(\ell+1) P_\ell(\tau) \right], \tag{9}$$

where $\tau \equiv \cos\theta$. The theorem can now be easily verified for this special case since, for $\theta = \pi$, the quantity in square brackets vanishes for every value of ℓ, and $\underset{\sim}{A}(\pi)$ is therefore identically zero.

When $ka << 1$, where a is the characteristic dimension of the scatterer, only the $\ell = 1$ phase shift is important, and one finds from Eq. (9) that

$$\left| \underset{\sim}{A}(\theta, \phi) \right|^2 = \frac{9}{4k^2} \sin^2\delta_1 (1 + \cos\theta)^2 ,$$

in contrast to a $(1 + \cos^2 \Theta)$ angular dependence of the differential cross-section when $\mu = 1$.[5] Thus, at least in the long wave length limit, the distribution shifts to predominantly forward scattering. Whether this is true also at higher frequencies is not known. However, for sufficiently short wavelengths the Schiff high-energy approximation for large-angle electromagnetic scattering[6] can, in principle, be used to compute the angular distribution in the neighborhood of the backward direction.

It is interesting to note that the Schiff formula also yields zero for the scattered amplitude in the backward direction when $\epsilon(\underset{\sim}{r}) = \mu(\underset{\sim}{r})$, under no assumptions other than $\epsilon\mu - 1 << 1$ and $kR >> 1$, where R is a characteristic dimension of the scatterer. The assumption of axial symmetry is not required; thus the theorem should be approximately valid for an arbitrary scatterer, provided only that $\epsilon = \mu \approx 1$, $kR >> 1$.

V. ACKNOWLEDGEMENTS

The authors wish to acknowledge several helpful discussions with Mr. Robert Margulies and Dr. Lee Frantz.

REFERENCES

1. A special case of this result was, to the authors' knowledge, first asserted by V. H. Weston (Conductron Corp. internal communication, 1961) who required $\epsilon = \mu =$ const. in the scattering medium.

2. This is true in Gaussian units, but because of the linearity of the equations we may make the more general statement that $\mathbf{H}(\mathbf{r}) = h\mathcal{R}\mathbf{E}(\mathcal{R}^{-1}\mathbf{r})$, where h is a constant appropriate to the chosen system of units.

3. D. Arnush, Space Technology Laboratory Report No. 6110-7466-RU-001, (unpublished).

4. W. K. H. Panofsky and M. Phillips, Classical Electricity and Magnetism, (Addison-Wesley Publishing Co., Reading, Mass., 1955), p. 205.

5. The result for $\mu = 1$ refers to the less general case of a homogeneous sphere, in which the differential cross-section has been integrated over ϕ. See J. A. Stratton, Electromagnetic Theory (McGraw-Hill Book Co., New York, 1941), p. 570.

6. L. I. Schiff, Phys. Rev. **104**, 1481 (1956).

UNCLASSIFIED

Space Technology Laboratories, Inc., One Space Park, Redondo Beach, California.
A THEOREM ON ELECTROMAGNETIC BACKSCATTER, by R. J. Wagner and P. J. Lynch. December 1962
9 p. (6110-7892-RU-000);
BSD-TDR-62-375
(Contract No. AF 04(694)-1 Unclassified Report

A theorem is proved which gives sufficient conditions under which electromagnetic backscatter from an inhomogeneous object vanishes identically.

UNCLASSIFIED

UNCLASSIFIED

Space Technology Laboratories, Inc., One Space Park, Redondo Beach, California.
A THEOREM ON ELECTROMAGNETIC BACKSCATTER, by R. J. Wagner and P. J. Lynch. December 1962
9 p. (6110-7892-RU-000);
BSD-TDR-62-375
(Contract No. AF 04(694)-1 Unclassified Report

A theorem is proved which gives sufficient conditions under which electromagnetic backscatter from an inhomogeneous object vanishes identically.

UNCLASSIFIED

UNCLASSIFIED

Space Technology Laboratories, Inc., One Space Park, Redondo Beach, California.
A THEOREM ON ELECTROMAGNETIC BACKSCATTER, by R. J. Wagner and P. J. Lynch. December 1962
9 p. (6110-7892-RU-000);
BSD-TDR-62-375
(Contract No. AF 04(694)-1 Unclassified Report

A theorem is proved which gives sufficient conditions under which electromagnetic backscatter from an inhomogeneous object vanishes identically.

UNCLASSIFIED

UNCLASSIFIED

Space Technology Laboratories, Inc., One Space Park, Redondo Beach, California.
A THEOREM ON ELECTROMAGNETIC BACKSCATTER, by R. J. Wagner and P. J. Lynch. December 1962
9 p. (6110-7892-RU-000);
BSD-TDR-62-375
(Contract No. AF 04(694)-1 Unclassified Report

A theorem is proved which gives sufficient conditions under which electromagnetic backscatter from an inhomogeneous object vanishes identically.

UNCLASSIFIED

Space Technology Laboratories, Inc., One Space Park, Redondo Beach, California.
A THEOREM ON ELECTROMAGNETIC BACKSCATTER, by R. J. Wagner and P. J. Lynch. December 1962
9 p. (6110-7892-RU-000);
BSD-TDR-62-375
(Contract No. AF 04(694)-1 Unclassified Report

A theorem is proved which gives sufficient conditions under which electromagnetic backscatter from an inhomogeneous object vanishes identically.

UNCLASSIFIED

UNCLASSIFIED

Space Technology Laboratories, Inc., One Space Park, Redondo Beach, California.
A THEOREM ON ELECTROMAGNETIC BACKSCATTER, by R. J. Wagner and P. J. Lynch. December 1962
9 p. (6110-7892-RU-000);
BSD-TDR-62-375
(Contract No. AF 04(694)-1 Unclassified Report

A theorem is proved which gives sufficient conditions under which electromagnetic backscatter from an inhomogeneous object vanishes identically.

UNCLASSIFIED

UNCLASSIFIED

Space Technology Laboratories, Inc., One Space Park, Redondo Beach, California.
A THEOREM ON ELECTROMAGNETIC BACKSCATTER, by R. J. Wagner and P. J. Lynch. December 1962
9 p. (6110-7892-RU-000);
BSD-TDR-62-375
(Contract No. AF 04(694)-1 Unclassified Report

A theorem is proved which gives sufficient conditions under which electromagnetic backscatter from an inhomogeneous object vanishes identically.

UNCLASSIFIED

UNCLASSIFIED

Space Technology Laboratories, Inc., One Space Park, Redondo Beach, California.
A THEOREM ON ELECTROMAGNETIC BACKSCATTER, by R. J. Wagner and P. J. Lynch. December 1962
9 p. (6110-7892-RU-000);
BSD-TDR-62-375
(Contract No. AF 04(694)-1 Unclassified Report

A theorem is proved which gives sufficient conditions under which electromagnetic backscatter from an inhomogeneous object vanishes identically.

UNCLASSIFIED

UNCLASSIFIED